The Taming of the Gyrator

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Abstract

The gyrator — or simulated inductor — is a valuable component for use in low-frequency electronic filters. It is economical, and offers potentially excellent performance over a frequency range where conventional inductors are difficult to realise.

Unfortunately, the conventional circuit configuration, involving two ‘nose-to-tail’ operational amplifiers, is very susceptible to high-frequency oscillations. This report analyses the circuit in detail, and proposes a modification to overcome the problem.

The BBC has been granted a patent, GB2365235, for the modification.

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1. Introduction

The gyrator — or simulated inductor — is a valuable component for use in low-frequency electronic filters. It is economical to implement, and offers potentially excellent performance at low frequencies where conventional inductors are difficult to realise.

Unfortunately, the conventional circuit configuration, which includes two operational amplifiers in a nose-to-tail feedback arrangement, is very susceptible to high-frequency oscillations. This report analyses the problem and explains how to design gyrator circuits successfully.

2. How It Works

Firstly, consider the behaviour of a ‘real’ inductor possessing inductance $L$. If a current $I$ is passing through it, a voltage $V$ will appear across it, as defined by the following formula:

$$V = L \frac{dI}{dt};$$

where $dI/dt$ is the rate of change of $I$ with respect to time. It follows that, if a fixed voltage $V_0$ is applied across the inductor,

$$I = \frac{(V_0/L)}{\int dt},$$

or $(V_0/L) t$.

In other words, the current through the inductor increases linearly with time.

Now consider the gyrator circuit shown below:

![Figure 2.1: Gyrator Circuit with a Voltage Step Applied to It](image)

Despite its eccentric appearance, the gyrator is easy to understand. At rest, when the applied voltage $V$ is zero, all voltages within the circuit are also zero. If $V$ suddenly changes from zero to $V_0$, all the circuit nodes instantly follow suit. However, this is not a stable condition, because a voltage exists across $R_4$ and the resulting current must charge $C_1$. As $C_1$ charges, the voltage across $R_4$ — and hence the output of $A_2$ — tends to fall. $A_1$ and resistors $R_2$ and $R_3$ form a unity-gain inverting amplifier which acts to oppose the fall in voltage across $R_4$. Hence the current through $R_4$ remains at a constant value of $V_0/R_4$, and for this to be true the output of $A_1$ must rise at a rate equal to the current divided by the value of $C_1$; in other words

$$V_{A1\text{ output}} = V_0 + \frac{(V_0/C_1.R_4)}{(t - t_0)}.$$  

(The voltage step is applied at time $t_0$, at which instant $V_{A1\text{ output}}$ rises to $V_0$.) Feedback resistors $R_2$ and $R_3$ ensure that the output of $A_2$ falls at exactly the same rate; or

$$V_{A2\text{ output}} = V_0 - \frac{(V_0/C_1.R_4)}{(t - t_0)}.$$
The difference between the applied voltage and that at the output of A2 determines the current flowing through R1:

\[ I = \frac{(V_0 - V_{A2\, output})}{R_1}, \text{ so} \]
\[ I = \frac{(V_0/C1.R4.R1)}{(t - t_0)}. \]

Comparison of this formula with the earlier one for the ‘real’ inductor shows that the gyrator behaves as an inductor of value C1.R4.R1. The convention, when designing gyrator circuits, is to make all four resistor values equal. However, the only requirement is for R2 to equal R3.

3. An Example of a Filter Design Using Gyrators

In principle, designing filters with gyrators is simple, because it is usually possible to regard the gyrator as a nearly ideal component. Reference 1 states that Q values of about 150 are practicable — a performance that would be hard to achieve with a wound component at audio frequencies. Furthermore, the ferrites used to make such ‘real’ inductors tend to possess significant temperature coefficients.

The example filter was required to have a 3 dB bandwidth of about 250 Hz centred on 15.625 kHz. A second-order response was deemed to provide adequate out-of-band attenuation — approximately 30 dB at 750 Hz from the centre frequency. A 0.1 dB Chebyshev filter was selected as the optimum compromise between in-band response flatness and out-of-band rejection. Table 5.5 of Reference 1 provides the following coefficients:

<table>
<thead>
<tr>
<th>R_L</th>
<th>R_S</th>
<th>C12</th>
<th>L1</th>
<th>C2</th>
<th>L2</th>
<th>R_c</th>
</tr>
</thead>
</table>

Normalised resonator Q-factors:
\[ q_1 = q_2 = 1.638 \]

Coupling coefficient:
\[ k_{12} = 0.711 \]

Figure 3.1: Realisation of the Bandpass Filter

If the actual resonator Q is chosen as 100, the filter Q \((Q_{bp})\) is given by

\[ Q_{bp} = \frac{Q}{q_1} = \frac{100}{1.638}, \]

or

\[ Q_{bp} = 61.05. \]

The 3 dB bandwidth of the filter \((BW_{3dB})\) is then given by

\[ BW_{3dB} = f_0 / Q_{bp}, \text{ where } f_0 \text{ is the centre frequency,} \]
\[ = 15,625 / 61.05; \]
\[ so \]
\[ BW_{3dB} = 256 \text{ Hz.} \]

Outside the 3 dB bandwidth, the response of an \(n^{th}\)-order filter at frequency \(f\) falls approximately as follows:

\[ |A| = \left( \frac{BW_{3dB}/2}{(f - f_0)} \right)^n. \]

\[ ^1 \text{Chebyshev filters have the disadvantage of providing a poor transient response, especially with high orders. In this case, the transient response was of little consequence, because the signal would eventually pass through an even narrower filter. The characteristics of the second filter would dominate.} \]

---

1 Chebychev filters have the disadvantage of providing a poor transient response, especially with high orders. In this case, the transient response was of little consequence, because the signal would eventually pass through an even narrower filter. The characteristics of the second filter would dominate.
If \( n = 2 \) and \( BW_{3dB} = 256 \) Hz, the response at 750 Hz from \( f_0 \) will be

\[
|A| = \left( \frac{256/2}{750} \right)^2.
\]

Hence the response is about 1/30, corresponding to the required 30 dB rejection.

It would be possible to choose any value of \( R_S \) and \( R_L \) and calculate the remaining circuit components accordingly. However, the gyrator configuration imposes practical limits. For instance, the resistor values (\( R_1 \) and \( R_4 \) in Figure 2.1) should be about 1 kΩ, whilst the capacitor value (\( C_1 \)) should not exceed 1 \( \mu \)F. Assuming for the moment that \( f_0 \) corresponds to \( \omega_0 = 10^5 \), the inductive reactance offered by the gyrator (\( \omega_0 L \)) is given by

\[
X_L = \omega_0 C_1 R_4 R_1 = 10^5 \cdot C_1 \cdot 10^3 \cdot 10^3.
\]

\( R_4 \) and \( R_1 \) have been taken as 1 kΩ. If \( X_L \) is also taken as 1 kΩ, \( C_1 \) becomes \( 10^{-8} \) F or 10 nF, whilst \( R_S \) and \( R_L \) become 1 kΩ \( \times Q \) or 100 kΩ. These are sensible values.

At resonance, the nodal capacitance \( C_{node} \) (\( C_1 \) and \( C_2 \) approximately in Figure 3.1) must also have a reactance of 1 kΩ. Hence \( C_{node} \) equals 10 nF.

The value of coupling capacitor \( C_{12} \) is given by

\[
C_{12} = C_{node} \left( k_{12} / Q_{bp} \right) = 10^{-8} \cdot (0.711 / 61.05) ;
\]

hence

\[
C_{12} = 116.5 \text{ pF}.
\]

Finally, the value of the resonating capacitors \( C_1 \) and \( C_2 \) is given by

\[
C_1 = C_2 = C_{node} = C_{12}, \text{ or } 9.883 \text{ nF}.
\]

When the filter design was finalised, \( \omega_0 \) was given its true value of \( 2\pi \times 15,625 \), but \( C_{node} \) was retained as 10 nF precisely. The difference this made to the theoretical component values was very small; for instance, \( R_S \) and \( R_L \) became 101.86 kΩ rather than 100 kΩ. Such changes are masked in practice by component tolerances and the use of preferred values.

4. **The Problem**

The filter described above was built using MAX412 dual op-amps as the active circuit elements. Amongst their many virtues, these devices possess a 28 MHz gain-bandwidth product and are unity-gain stable. Although such performance might seem excessive for a filter working below 20 kHz, the gyrator relies on the op-amp performance being close to ‘ideal’. Unfortunately, when the filter was tested, the gyrators were evidently unstable. At first glance it appeared that an oscillation was occurring at 15.625 kHz, the natural frequency of the resonators. In fact, the oscillation was at about 1 MHz, but ‘squegging’ at the lower frequency. The resonators were then removing the high frequency component, to leave a ‘clean’ LF sinusoid at the output of the filter.

Some time was spent in attempting to remove the spurious oscillations. Conventional cures such as dabbing small value capacitors between likely points in the circuit were not very helpful. Once stopped, the oscillations were likely to return, particularly if a change was made to the filter design. Hence there was a need to study the problem in greater detail.

\[\text{\textsuperscript{2}}\text{ ‘Approximately’ because calculation of the exact value must take into account the coupling capacitor } C_{12}. \text{ Generally, the coupling capacitance is much less than the nodal capacitance, and the approximation is good.}\]
5. Analysing the Performance of the Gyrator

At first glance, there does not seem to be an easy way to analyse or measure the performance of the gyrator: the fortunes of the two op-amps appear hopelessly dependent on one another. However, a little ingenuity allows the circuit to be separated into two parts: an all-pass phase shifter and a unity-gain inverting amplifier:

![Diagram of the Gyrator](image)

**Figure 5.1:** ‘Deconstruction’ of the Gyrator

The first step is to divide the ‘see-saw’ feedback resistors into two, as shown in the left-hand diagram. If R2a and R2b are each made twice the value of the original R2, likewise R3a and R3b, the operation of the circuit is unchanged. It is now possible to separate the inverting inputs of A1 and A2, again without affecting the operation in any way.

The right-hand diagram shows that the two separated parts of the gyrator can be thought of as forming a loop. If the loop is broken at a convenient point, the performance of each part — or the two parts in tandem — can be measured without difficulty. Note that the original input is shown as being grounded. This is a reasonable simplification to make when considering the circuit behaviour at high frequencies, as a capacitor would normally be connected across the input.\(^3\) Once the input has been grounded, R1 is redundant.

Assuming that A1 and A2 are ‘ideal’ op-amps of infinite gain and no phase shift, the all-pass phase shifter provides a gain of +1 at high frequencies, whilst the inverter has a gain of –1. The complete loop has a gain of –1 and is therefore stable. A ‘real’ op-amp must possess a response which falls with frequency, and this falling response is associated with a phase shift. When feedback is applied the op-amp — as it is in each part of the gyrator — the result may be a high frequency gain peak as well as a phase shift. Should the phase shift within the complete gyrator loop equal 180° whilst the gain is greater than unity, the gyrator will oscillate.

Appendix 1 shows that op-amps using dominant pole compensation should be stable in a gyrator configuration. If such well-behaved op-amps are used, neither part of the gyrator can ever have a gain of greater than unity and oscillation is not possible. However, as the laboratory measurements described in the next section show, real life is not that kind.

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\(^3\) Of course, this is resonating capacitor C1 or C2 in Figure 3.1.
6. Laboratory Measurements on the All-pass Phase Shifter

Firstly, gain and phase measurements were made on the all-pass phase shifter. Since it would be necessary to explore frequencies up to about 100 MHz, a network analyser was chosen as the test equipment. The MAX412 op-amp cannot drive a 50 Ω termination comfortably, so a 450 Ω resistor was added in series with its output. During the measurements, an allowance was made for the 20 dB loss introduced by this.

![Practical All-pass Phase Shifter Circuit](image)

Figure 6.1: Practical All-pass Phase Shifter Circuit

The plot below confirms the calculations given in Appendix 1: the phase shift is +180° at low frequencies, and falls to +90° at a frequency corresponding to the time-constant 1 kΩ × 10 nF. As the frequency is increased, the phase shift continues to fall towards its theoretical limiting value of 0°, which it reaches at about 1 MHz. Up to this point, the gain is substantially constant.

![Measured Gain and Phase Responses of the All-pass Phase Shifter](image)

Figure 6.2: Measured Gain and Phase Responses of the All-pass Phase Shifter

Beyond 1 MHz, the effects of the limited op-amp bandwidth become apparent. As the frequency is increased still further, the phase shift reaches −90° at about 13 MHz, and the gain falls below unity shortly afterwards. Once again, this agrees well with the theory in Appendix 1, which shows that the bandwidth of the phase shifter should equal half the gain-bandwidth product of the op-amp. Unfortunately, there is a significant gain peak of 4 dB at 10 MHz, which implies that the op-amp itself is under-compensated; that is, its phase shift exceeds −90° at the unity-gain point.

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4 Perhaps something should have been said about conventions at this point. A negative value of phase shift is taken to imply a phase delay. Note that, where the text mentions the phase shift exceeding −90°, for instance, it really means the modulus of the phase shift exceeding 90°. Although the terminology is not strictly correct, it is unlikely to cause confusion.
The reason for the additional phase shift is worth exploring. The plot overleaf shows the effect of removing R2 and changing R1 to 1 kΩ, so transforming the amplifier into a unity-gain buffer at high frequencies. (The change to R1 was made to keep the impedance ‘seen’ by the inverting input the same as before.)

![Graph showing the effect of changing R1 to 1 kΩ](image)

**Figure 6.3:** Measured Gain and Phase Responses of the Unity-gain Buffer

As expected, the bandwidth now exceeds 20 MHz, but there is a considerable gain peak of about 9 dB. The gain peak can be reduced substantially by replacing R1 with a short-circuit, implying that the problem is partly caused by the input capacitance of the op-amp. Removal of the offending resistor is not possible in a gyrator, but it is obviously helpful to make the feedback resistors as low a value as practicable.

A possible improvement is to reduce the open-loop gain of the op-amp. The plot below shows the performance of the all-pass phase shifter when a 220 Ω resistor is placed between the inputs of the op-amp:

![Graph showing the effect of reducing the open-loop gain](image)

**Figure 6.4:** Effect of Reducing the Open-loop Gain of the Op-amp
The gain peak has disappeared, but the phase response has suffered because of the reduction in loop bandwidth. Unfortunately, as Appendix 1 shows, the gyrator is sensitive to phase errors.

In order to reduce the effect of the resistor at frequencies within the loop bandwidth, a capacitor may be added in series with the resistor; that is, a Zobel network may be connected across the op-amp inputs:

![Figure 6.5: Addition of a Zobel Network](image)

Although this idea is effective, the designer should remember that the loop gain now falls at 12 dB per octave over a limited frequency range. Hence the circuit is only conditionally stable — a situation that is perhaps best avoided.

### 7. Laboratory Measurements on the Unity-gain Inverting Amplifier

Secondly, the gain and phase measurements were repeated on the unity-gain inverting amplifier; the test arrangements were just as they were before:

![Figure 7.1: Practical Unity-gain Inverting Amplifier Circuit](image)

![Figure 7.2: Measured Gain and Phase Responses of the Unity-gain Inverting Amplifier](image)
At first sight, the unity-gain inverting amplifier looks slightly ‘tamer’ than the phase shifter: the
gain peak is 3 dB at 10 MHz, 1 dB less than before. Also, the bandwidth is approximately half
the 28 MHz gain-bandwidth product of the op-amp, in agreement with the calculations given in
Appendix 1. Unfortunately, the phase response is particularly ‘unfriendly’, reaching nearly –90°
at the frequency of the gain peak and continuing to increase rapidly thereafter.

It seems that, if it is necessary to reduce the high-frequency gain of either the phase shifter or the
inverting amplifier, the inverting amplifier would be the better choice. The various ways of
damping the phase shifter all serve to increase the phase lag at high frequencies, whereas it would
be difficult to make the inverting amplifier much worse. As a demonstration, a 100 pF capacitor
was added between the output of the op-amp and its inverting input. The result should be a
response falling at 6 dB per octave beyond \( \omega = \frac{1}{2 \, \text{k}\Omega \times 100 \, \text{pF}} \), a frequency of about
800 kHz (note the change of vertical scale):

![Diagram](image)

**Figure 7.3:** Effect of Adding a Compensation Capacitor to the Feedback Resistor

The results are much as predicted: at 800 kHz, the gain is –3 dB and the phase shift –45°; at
higher frequencies, the gain falls at 6 dB per octave and the phase shift approaches –90°.
However, note that beyond about 14 MHz the phase shift is less than it was before the
compensation was included. Of course, this amount of compensation is more than would
normally be used in practice.

8. Laboratory Measurements on the Complete Loop

Finally, similar gain and phase measurements were made on the unity-gain inverting amplifier
and all-pass phase shifter in tandem. As the plot overleaf shows, the gain of the uncompensated
combination is greater than 6 dB when the phase shift of the complete loop reaches –360°.5 (The
corresponding frequency is about 12 MHz.) It is not surprising that the combination oscillates
when the loop is completed! It would be necessary to reduce the loop gain by around 10 dB at
12 MHz, without increasing the phase shift, to ensure a safe gain margin.

---

5 That is, when the unwanted phase shift introduced by the unity-gain inverting amplifier and all-pass phase shifter
reaches –180°.
Figure 8.1: Practical Inverting Amplifier and All-pass Phase Shifter in Tandem

Figure 8.2: Measured Gain and Phase Responses of the Uncompensated Circuit

The last step is to add the compensation capacitor as described in the previous section. In fact, the capacitor was reduced to 27 pF, corresponding to a –3 dB point of about 3 MHz. At 12 MHz, where the total loop phase shift has reached –360°, the gain will have been reduced by 12 dB. Since the all-pass phase shifter introduces a gain peak of 4 dB, the overall gain margin should be a healthy 8 dB. The plot overleaf confirms this.

If the best possible performance is wanted from the gyrator, the compensation capacitor should be made even smaller. As the next section demonstrates, there is a penalty to pay: the small phase shift introduced at the nominal working frequency of the gyrator (about 16 kHz in this example) adds a significant resistive component. The effective Q of the inductor is therefore reduced. An allowance can be made during the design of the associated filter, much as would be done if ‘real’ inductors were being used; but it makes sense to keep the Q as high as reasonably possible.
9. The Performance of the Gyrator

To check the performance in a practical application, the gyrator was used as the inductive element in a parallel tuned circuit:

With the same component values as before, the inductance of the gyrator is $R^2C$, or $1\,\text{k}\Omega \times 1\,\text{k}\Omega \times 10\,\text{nF}$, which equals $10\,\text{mH}$. If the resonating capacitor is made equal to $C$, the resonant frequency is given by

$$f_0 = \frac{1}{2\pi} \times \sqrt{\frac{1}{(R^2C \times C)}} = \frac{1}{2\pi} \times \frac{1}{(RC)}$$

or about $16\,\text{kHz}$. The $200\,\text{k}\Omega$ resistors together shunt the tuned circuit with a parallel resistance of $100\,\text{k}\Omega$, to give a quality factor

$$Q = \frac{(\text{parallel resistance})}{|\text{impedance of gyrator at } f_0|} = \frac{100\,\text{k}\Omega}{2\pi f_0 \times R^2C} = \frac{10^5}{(10 \times 10^{-2})}$$

or 100. The expected $–3\,\text{dB}$ bandwidth of the tuned circuit should therefore be $16\,\text{kHz}/100$, or $160\,\text{Hz}$. The plot overleaf shows the actual results obtained.
The circuit is evidently well-behaved, but its –3 dB bandwidth is about 320 Hz, or twice its calculated value. Part of the reason for this is the small phase lag introduced by the compensation capacitor. Appendix 2 shows that this causes a resistive component to appear in series with the synthesised inductance, hence reducing its $Q$ to $Q'$:

$$Q' = \frac{2}{\omega_1/\omega_0},$$

where $\omega_0$ is the resonant frequency of the tuned circuit, and $\omega_1$ is the –3 dB point introduced by the compensation capacitor. In the present example, where $\omega_0$ equals $10^5$ and $\omega_1$ equals $1/(2 \times 27 \times 10^{-12})$ pF),

$$Q' = \frac{2}{(2.10^3 \times 27.10^{-12}) (10^5)},$$

or about 370.

The measured performance of the tuned circuit suggests that the actual value of $Q'$ is closer to 100. However, there are other losses that need to be taken into account. In particular, the polyester capacitor (unwisely) used as the tuning element is quoted as having a power factor of 0.008 at 1 kHz. This in itself would reduce the $Q$ to 125.

Appendix 2 also shows that the apparent value of the inductance is changed by a fraction corresponding to $1/Q'$; that is, the modified inductance $L'$ is given by

$$L' = L \left(1 + \frac{1}{Q'}\right),$$

where $L$ is the inductance offered by a ‘perfect’ gyrator. The change is likely to be insignificant in practice, as it can be absorbed in the adjustment that must be included to cater for component variations.

When designing a filter, it would make sense to calculate $Q'$ and make an appropriate allowance. However, where the performance is critical, final adjustments should be made ‘at the bench’.

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6 Strictly, $\omega_0$ is the frequency at which the all-pass phase shifter introduces $90^\circ$ phase lead. However, this is usually chosen as the working point of the gyrator.
10. Summary and Recommendations for Gyrator Design

This report has discussed the design and performance of gyrator circuits in detail. Practical measurements have shown that good performance is achievable, but that precautions might be necessary to avoid instability. A modification to the basic circuit has been suggested which, correctly proportioned, will ensure high-frequency stability. It is recommended that the following should be borne in mind when designing a gyrator circuit:

- The op-amps should possess a large gain-bandwidth product. If possible, this should be an order of magnitude greater than the operating frequency ($\omega_0 / 2\pi$) multiplied by the desired quality factor (Q).
- A dual op-amp package should always be used in a gyrator circuit; otherwise DC offsets could present a problem.\(^7\)
- If the op-amps possess pure dominant pole compensation, they should be stable if used in a basic gyrator configuration.
- However, an op-amp with a large gain-bandwidth product is likely not to possess such ideal compensation. Because of this, its gain can exhibit a significant high-frequency peak. A pair of such op-amps will almost certainly be unstable in the basic gyrator.
- Gain peaking can be reduced by (1) making the value of the op-amp feedback resistors as small as possible; (2) keeping the op-amp input capacitance to a minimum. The smallest value of the resistors is determined by the loading imposed on the op-amp outputs, whilst the input capacitance is minimised by careful circuit-board design.
- Stability can be achieved by splitting the feedback resistors to the op-amp pair, so that the high-frequency responses of the two can be tailored individually. It is suggested that a time-constant ($1/\omega_1$) is introduced into the ‘unity-gain inverting amplifier’, so creating a dominant lag within the gyrator.
- ($1/\omega_1$) needs to be determined by practical measurement. It should be chosen such that the gain within the gyrator loop has fallen well below unity once the excess loop phase shift reaches –180°.
- Where high performance is required from a filter making use of gyrators, allowance must be made for the damping introduced by the dominant lag compensation. The maximum achievable quality factor $Q'$ equals ($2\omega_1 / \omega_0$).

11. Acknowledgements

The author would like to thank the following: Malcolm Williams for building the prototype gyrator; Pete Moss for his encouragement and theoretical insights; John Salter and Adrian Robinson for checking the text of this report — and correcting the mathematics where necessary!

12. References


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\(^7\) This subject has not been considered in the text. A moment’s thought will show that, if the two op-amps have different input offset voltages, the common DC feedback arrangements cannot keep both op-amps ‘happy’ at the same time! It might be tempting to overcome the problem by AC coupling the all-pass phase shifter to the unity-gain inverting amplifier. However, the loop phase shift approaches 0° at low frequencies, and the extra time-constant would certainly result in LF instability.
Appendix 1. Analysis of Gyrator Performance

This appendix examines the high-frequency performance of the two essential ingredients of the gyrator: the all-pass phase shifter and the unity-gain inverter. Firstly, the all-pass phase shifter:

\[
\begin{align*}
\text{Figure A1.1: All-pass Phase-shift Circuit}
\end{align*}
\]

\[
v_1 = v_{\text{in}} \{ R / ( R + j\omega C ) \} = v_{\text{in}} \{ j\omega CR / ( j\omega CR + 1 ) \}, \quad \text{or} \\
v_1 = v_{\text{in}} \{ (j\omega/\omega_1) / ( j\omega /\omega_1 + 1 ) \}; \quad \text{.................................................. (1)}
\]

also \( v_2 = (v_{\text{in}} + v_{\text{out}}) / 2 \) .................................................. (2)

and \( v_{\text{out}} = (v_1 - v_2) A \). .................................................. (3)

Substituting (1) and (2) into (3):

\[
\begin{align*}
\text{At low frequencies, as } \omega \rightarrow 0, \quad v_{\text{out}} / v_{\text{in}} \rightarrow -1; \quad \text{at high frequencies, as } \omega \rightarrow \infty, \quad v_{\text{out}} / v_{\text{in}} \rightarrow +1. \\
\end{align*}
\]

That is, the circuit is a unity-gain inverting amplifier at low frequencies and a unity-gain non-inverting amplifier at high frequencies.

Rewriting (5) and multiplying numerator and denominator by \((1 - j\omega /\omega_1)\):

\[
\begin{align*}
(v_{\text{out}} / v_{\text{in}}) &= 2 (j\omega /\omega_1) - (1 + j\omega /\omega_1) \\
&= -(1 - j\omega /\omega_1)(1 - j\omega /\omega_1) \\
&= -\left(1 - (\omega /\omega_1)^2 \right)^2 \\
\end{align*}
\]

Separating this expression into real and imaginary parts:

\[
\begin{align*}
(v_{\text{out}} / v_{\text{in}}) &= -\left\{ 1 - (\omega /\omega_1)^2 \right\} - 2(j\omega /\omega_1) \\
\end{align*}
\]

The magnitude of this, \( |v_{\text{out}} / v_{\text{in}}| \), is given by \( \sqrt{\text{Real}^2 + \text{Imag}^2} \):

\[
\begin{align*}
|v_{\text{out}} / v_{\text{in}}|^2 &= \left\{ 1 - 2(\omega /\omega_1)^2 + (\omega /\omega_1)^4 \right\} + 4(\omega /\omega_1)^2, \\
&= \left\{ 1 + (\omega /\omega_1)^2 \right\}^2 \\
\end{align*}
\]

so \( |v_{\text{out}} / v_{\text{in}}|^2 = \left\{ 1 + (\omega /\omega_1)^2 \right\}^2 \).
\[
\left\{ \frac{1 + (\omega/\omega_1)^2}{1 - (\omega/\omega_1)^2} \right\}^2
\]

Therefore \( |v_{out}/v_{in}| = 1 \); so the circuit always has unity gain.

The argument of this, \( \phi(v_{out}/v_{in}) \), is given by \( \tan^{-1} \left( \text{Imag} / \text{Real} \right) \):

\[
\tan \phi(v_{out}/v_{in}) = -\frac{2\omega/\omega_1}{1 - (\omega/\omega_1)^2} . \quad \text{........................................... (6)}
\]

The phase shift is therefore \(+\pi\) radians at very low frequencies and 0 at very high frequencies; it is \(\pi/2\) when \(\omega\) equals \(\omega_1\).

Now consider what happens at high frequencies when the op-amp is not ideal; that is, when \(A\) is not infinite. Most op-amps have a response which approximates to that of an integrator: the gain is very large at low frequencies, and falls at a rate of 6 dB per octave:

\[
A = \omega_2/j\omega; \quad |A| = 1 \text{ when } \omega = \omega_2 . \quad \text{........................... (7)}
\]

\(\omega_2/2\pi\) is a constant known as the gain-bandwidth product. The presence of \(j\) indicates that the phase shift associated with \(A\) is always \(-\pi/2\).

Returning to Equation (4) and inserting (7):

\[
(v_{out}/v_{in}) (2/A + 1) = 2 \left\{ \left( j\omega/\omega_1 \right) / \left( j\omega/\omega_1 + 1 \right) \right\} - 1 .
\]

\[
(v_{out}/v_{in}) (2j\omega/\omega_2 + 1) = 2 \left\{ \left( j\omega/\omega_1 \right) / \left( j\omega/\omega_1 + 1 \right) \right\} - 1 .
\]

Assuming \(\omega\) is much greater than \(\omega_1\), and that \(\omega_2\) is much greater than \(\omega_1\):

\[
(v_{out}/v_{in}) (2j\omega/\omega_2 + 1) = 1 ;
\]

\[
(v_{out}/v_{in}) = 1 / (2j\omega/\omega_2 + 1) = \omega_2 / (\omega_2 + 2j\omega) \quad \text{........................... (8)}
\]

Rewriting (8) and multiplying numerator and denominator by \((\omega_2 - 2j\omega)\):

\[
(v_{out}/v_{in}) = \omega_2 (\omega_2 - 2j\omega) / (\omega_2 + 2j\omega) (\omega_2 - 2j\omega)
= (\omega_2^2 - 2j\omega \omega_2) / (\omega_2^2 + 4\omega^2) .
\]

Therefore, \( |v_{out}/v_{in}|^2 = \omega_2^2 (\omega_2^2 + 4\omega^2) / (\omega_2^2 + 4\omega^2)^2 \), and

\[
|v_{out}/v_{in}| = \omega_2 / (\omega_2^2 + 4\omega^2)^{1/2} ; \quad \text{.............................. (9)}
\]

and \( \tan \phi(v_{out}/v_{in}) = -2\omega / \omega_2 \). \quad \text{.............................. (10)}

Equations (9) and (10) show that at \(\omega = \omega_2/2\) the phase shift is \(-\pi/4\) and the gain is \(-3\) dB. If the amplifier had been connected as a simple unity gain,

\[
\begin{align*}
\text{Figure A1.2: Unity-gain Amplifier Circuit}
\end{align*}
\]

the corresponding frequency would have been \(\omega_2\). The all-pass phase-shift circuit has half the bandwidth of the unity-gain amplifier.
Secondly, the unity-gain inverting amplifier:

\[ v_1 = 0 ; \] ................................. (11)
\[ v_2 = ( v_{\text{in}} + v_{\text{out}} ) / 2 ; \] ................................. (12)

and
\[ v_{\text{out}} = ( v_1 - v_2 ) A . \] ................................. (13)

Substituting (11) and (12) into (13):
\[ v_{\text{out}} = -A ( v_{\text{in}} + v_{\text{out}} ) / 2 , \text{ so} \]
\[ v_{\text{out}} ( 1 + A/2 ) = -v_{\text{in}} A/2 \]
\[ (v_{\text{out}}/v_{\text{in}}) = -A / ( 2 + A ) \] ................................. (14)

Assume \( A = \infty \); then
\[ (v_{\text{out}}/v_{\text{in}}) = -1 . \] ................................. (15)

Now consider what happens at high frequencies when the op-amp is not ideal; that is, when \( A \) is not infinite. Taking
\[ A = \omega_2 / j\omega , \text{ as before:} \]
\[ (v_{\text{out}}/v_{\text{in}}) = - (\omega_2 / j\omega ) / ( 2 + (\omega_2 / j\omega ) ) \]
\[ = - (\omega_2 ) / ( \omega_2 + 2j\omega ) \]

Rewriting this and multiplying numerator and denominator by ( \( \omega_2 - 2j\omega \) ):
\[ (v_{\text{out}}/v_{\text{in}}) = - \omega_2 ( \omega_2 - 2j\omega ) / \{ ( \omega_2 + 2j\omega ) ( \omega_2 - 2j\omega ) \} \]
\[ = - ( \omega_2^2 - 2\omega_2 \omega + 4\omega^2 ) / ( \omega_2^2 + 4\omega^2 ) . \]

This is just the same expression as was obtained for the all-pass phase shifter, except for the minus sign. Therefore,
\[ |(v_{\text{out}}/v_{\text{in}})| = \omega_2 / ( \omega_2^2 + 4\omega^2 )^{1/2} ; \] ................................. (16)
and \[ \tan \varphi(v_{\text{out}}/v_{\text{in}}) = -2\omega / \omega_2 . \] ................................. (17)

The unity-gain inverting amplifier has half the bandwidth of the non-inverting unity-gain amplifier.

The above result is intuitively reasonable. If one imagines \( v_{\text{in}} \) being grounded, and the input signal being applied as \( v_1 \) instead, it is easy to see that the closed-loop gain will be 2 at low frequencies. Hence, for a given gain-bandwidth product, the bandwidth will be half that of a unity-gain amplifier.
Appendix 2. Effect of Compensating the Gyrator

This appendix analyses the gyrator circuit and examines the effect of including a ‘dominant lag’. The circuit and its associated equations are shown below:

\[ i_{in} = \frac{(v_{in} - v_5)}{R} \quad \text{............... (1)} \]

\[ v_1 = v_{in} \quad \text{............... (2)} \]

\[ v_3 = v_2 \frac{j\omega CR}{1 + j\omega CR} \quad \text{............... (3)} \]

\[ v_4 = v_3 \quad \text{............... (4)} \]

\[ v_5 - v_4 = v_4 - v_2 \quad \text{............... (5)} \]

\[ v_5 - v_1 = \frac{v_1 - v_2}{2R} \quad \text{............... (6)} \]

![Gyrator Circuit to Be Analysed](image)

Combining (1) and (5),
\[ i_{in} = \frac{(v_{in} + v_2 - 2v_4)}{R} ; \]

and including (4),
\[ i_{in} = \frac{(v_{in} + v_2 - 2v_3)}{R} . \]

Combining this with (3),
\[ i_{in} R = v_{in} + v_2 - 2v_2 \frac{j\omega CR}{1 + j\omega CR} ; \]

so
\[ i_{in} R = v_{in} + v_2 \left( 1 - j\omega CR \right) . \quad \text{........ (7)} \]

Eliminating \( v_5 \) from (1), using (2) and (6),
\[ i_{in} R = v_{in} - \left\{ v_{in} + 2R \left( \frac{v_{in} - v_2}{Z} \right) \right\} ; \]

so
\[ v_2 = v_{in} + i_{in} \frac{Z}{2} \quad \text{............... (8)} \]

Substituting (8) into (7),
\[ i_{in} R = v_{in} + (v_{in} + i_{in} \frac{Z}{2}) \left( 1 - j\omega CR \right) ; \]

therefore
\[ v_{in} \left\{ 1 + \left( 1 - j\omega CR \right) \right\} = i_{in} \left\{ R - (Z/2) \left( 1 - j\omega CR \right) \right\} , \]

and
\[ v_{in} \left\{ (1 + j\omega CR) + (1 - j\omega CR) \right\} = i_{in} \left\{ R (1 + j\omega CR) - (Z/2) (1 - j\omega CR) \right\} ; \]

therefore
\[ v_{in}/i_{in} = (1/2) \left\{ R(1 + j\omega CR) - (Z/2)(1 - j\omega CR) \right\} . \quad \text{............... (9)} \]

Note that, if \( Z = 2R \),
\[ v_{in}/i_{in} = j\omega CR^2 ; \]

so the circuit offers an impedance \( v_{in}/i_{in} \) equivalent to that of an inductor \( CR^2 \).

Now suppose that the complex impedance \( Z \) is comprised of a resistor \( 2R \) in parallel with a small capacitor \( C' \). Then
\[ Z = \frac{(2R/j\omega C')}{\{2R + (1/j\omega C')\}} \]

If \( 1/C'2R = \omega_1 \),
\[ Z = \frac{2R}{\{j\omega_1\} + 1} . \quad \text{......... (10)} \]

\[ Z = \frac{2R}{(j\omega_1\omega_1) + 1}. \quad \text{......... (11)} \]
Inserting (10) into (9) and calling $1/CR = \omega_0$,

$$v_{\text{in}}/i_{\text{in}} = \frac{(R/2) \left( 1 + j\omega/\omega_0 \right) - (R/2) \left\{ (1 - j\omega/\omega_0) / (1 + j\omega/\omega_1) \right\}}{(1 + j\omega/\omega_1)}$$

$$= \frac{(R/2) \left\{ (1 + j\omega/\omega_0) (1 + j\omega/\omega_1) - (1 - j\omega/\omega_0) \right\}}{(1 + j\omega/\omega_1)}$$

$$= \frac{(R/2) \left\{ 1 + j\omega/\omega_0 + j\omega/\omega_1 - (\omega/\omega_0) (\omega/\omega_1) - 1 + j\omega/\omega_0 \right\}}{(1 + j\omega/\omega_1)};$$

$$= \frac{(R/2) \left\{ 2(\omega/\omega_0) + j\omega/\omega_1 - (\omega/\omega_0) (\omega/\omega_1) \right\}}{(1 + j\omega/\omega_1)}.$$

Multiplying numerator and denominator by $(1 - j\omega/\omega_1)$:

$$v_{\text{in}}/i_{\text{in}} = \frac{(R/2) \left\{ 2(j\omega/\omega_0) + j\omega/\omega_1 - (\omega/\omega_0) (\omega/\omega_1) \right\}}{(1 + j\omega/\omega_1)} 
\frac{(1 - j\omega/\omega_1)}{1 + (\omega/\omega_1)^2}$$

$$= \frac{(R/2) \left\{ 2(j\omega/\omega_0) + j\omega/\omega_1 - (\omega/\omega_0) (\omega/\omega_1) \right\}}{1 + (\omega/\omega_1)^2}$$

If $\omega_1 \to \infty$, $v_{\text{in}}/i_{\text{in}} \to R (j\omega/\omega_0)$ — an inductor of value $R/\omega_0$, or $R^2C$, as expected. Normally, $(\omega/\omega_1)^2 << 1$, so

$$v_{\text{in}}/i_{\text{in}} = \frac{(R/2) (j\omega/\omega_0) \left\{ 2 + \omega_0/\omega_1 \right\} + (R/2) \left\{ \omega^2/\omega_0 \omega_1 \right\}}{1 + (\omega/\omega_1)^2}.$$

(11)

The effect of the compensation is therefore to increase the inductance of the gyrator by a fraction $(1/2)(\omega_0/\omega_1)$. Generally this is a very small amount. If, as a typical example, $\omega_0$ is taken as $2\pi \times 16$ kHz, and $\omega_1$ as $2\pi \times 3$ MHz, the fractional increase is 1 part in 375. Such an error is easily absorbed by the adjustment that must be provided to accommodate normal component tolerances.

The compensation also introduces a resistive term $(R/2) \left\{ \omega^2/\omega_0 \omega_1 \right\}$. Although this appears negligible at first sight, it can represent an appreciable loss of $Q$. If the gyrator is used as part of a tuned circuit of resonant angular frequency $\omega_0$, the resistive component becomes $(R/2) \left\{ \omega_0/\omega_1 \right\}$. The $Q$ of the tuned circuit is given by

$$Q = \frac{\text{inductive impedance}}{\text{series resistance}} = \frac{(R) (\omega_0/\omega_0)}{(R/2) (\omega_0/\omega_1)};$$

or

$$Q = \frac{2 \omega_1/\omega_0}{\omega_0/\omega_1}.$$

(12)

In this example, the maximum possible $Q$ of the tuned circuit is 375. In practice, the $Q$ is likely to be less than this, because of other phase shifts within the gyrator and the power factor of the resonating capacitor.